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Abstract

Cox & Leland (2000) use techniques from the field of stochastic control theory to show that in the particular case of a Brownian motion for the asset returns all risk averse decision makers with a fixed investment horizon prefer path-independent pay-offs over path-dependent ones. We will provide a novel and simple proof for the Cox & Leland result and we will extend it to general, not necessarily complete, Lévy markets. It is also shown that in these markets optimal path-independent pay-offs have final values increasing with the underlying asset value. Our results imply that path-dependent investment pay-offs, the use of which is widespread in financial markets, do not appear to offer good value for risk averse decision makers with a fixed investment horizon.

1 Introduction

In this paper we analyse optimal investment choices in a Lévy market for risky assets. More precisely, let the risky asset price at time 0 be given by S_0. Then, we will assume that the stochastic process $\{X_t,\ t \geq 0\}$ is a Lévy process and we define the price S_t of the risky asset at time $t > 0$ as

$$S_t = S_0 e^{X_t}. \tag{1}$$

Lévy processes have proven to be successful in many areas of financial engineering such as equity, fixed income, commodities and recently also credit risk modelling. We will discuss their characteristics and properties in more detail in Section 2. For a full theoretical background we refer to Bertoin (1996) or Sato (2000). For more details about their applicability in finance we refer to Schoutens (2003).

We assume the market is frictionless, trading is continuous and there is a constant risk-free interest r. There are also no taxes, no transaction costs, no dividends, no restriction on borrowing or short sales and the risky asset is perfectly divisible. Finally, the notation $E_{\mathbb{P}}$ will be used to denote that expectations are taken with respect to a given (initial) physical probability measure $\mathbb{P}$.

We will consider an investor who is facing a fixed investment horizon of length $T > 0$, and who at time $t = 0$ is evaluating the appropriateness of a financial security with stochastic pay-off at time $t = T$ given by

$$P_g = g(S_{t_i} \mid 0 \leq t_i \leq T,\ i = 1, 2, ..., n), \tag{2}$$

for some function g. We will assume that g is such that $E_{\mathbb{P}}[P_g]$ exists. When the random variable P_g depends only on the final value S_T of the underlying risky asset then we call P_g a path-independent pay-off, otherwise P_g is path-dependent. The price (or cost) for buying the pay-off P_g will be denoted by $C(P_g)$.

Cox & Leland (2000) use techniques from the field of stochastic control theory to show that in the particular case of a Brownian motion for the log-returns risk averse decision makers with a fixed investment horizon prefer path-independent pay-offs over path-dependent pay-offs. We will provide a novel and simple proof for the Cox & Leland result and we will extend it to general, not necessarily complete, Lévy markets. Note that as compared to the use of a Brownian motion as the traditional workhorse for modelling log-returns, general Lévy processes provide more flexibility and potential accuracy. The latter holds especially true in case of short-term returns because they exhibit fat tails and auto-correlation; see e.g. Schoutens (2003).

This paper shows that for general Lévy markets path-independent pay-offs continue to be preferred by risk averse decision makers as long as the arbitrage-free pricing is based on the Esscher transform to generate an equivalent martingale measure.

Furthermore, we will show that in these instances investors with a fixed investment horizon $T > 0$ will always opt for path-independent pay-offs that are increasing with the underlying asset value S_T, a result that is related to earlier results of Dybvig (1988a,b).

Hence, we provide more support for the result that path-dependent pay-offs should always be avoided by risk averse utility maximisers, and they should buy path-independent structures instead. For example, click funds, which combine an investment guarantee with complicated path-dependent options to benefit from increasing stock markets, are of no real interest to investors.

The paper is structured as follows. In Section 2 we briefly recall some basic results from the field of Lévy processes, the ordering of risks, risk preferences, and we also discuss the Esscher transform as a tool to perform arbitrage-free pricing. In Section 3 we prove the optimality of path-independent investment strategies for Lévy processes and we give an example that allows explicit verification of our results. In Section 4 we show that an optimal path-independent pay-off is one where the pay-off values are increasing in the underlying asset

Distribution	$\nu(\mathrm{d}x)$
Poisson(λ)	$\lambda\delta(1)$
Normal(μ, σ^2)	0
Gamma(a, b)	$a \exp(-bx)x^{-1}1_{(x>0)}\mathrm{d}x$
IG(c, λ)	$(\pi)^{-1/2}cx^{-3/2}\exp(-\lambda x)1_{(x>0)}\mathrm{d}x$
VG(C, G, M)	$C\|x\|^{-1}(\exp(Gx)1_{(x<0)} + \exp(-Mx)1_{(x>0)})\mathrm{d}x$
NIG(α, β, δ)	$\delta\alpha\pi^{-1}\|x\|^{-1}\exp(\beta x)\mathrm{K}_1(\alpha\|x\|)\mathrm{d}x$
CGMY(C, G, M, Y)	$C\|x\|^{-1-Y}(\exp(Gx)1_{(x<0)} + \exp(-Mx)1_{(x>0)})\mathrm{d}x$
Meixner(α, β, δ)	$\delta x^{-1}\exp(\beta x/\alpha)\sinh^{-1}(\pi x/\alpha)\mathrm{d}x$

Table 1: Lévy measure for some Lévy Processes (at time $t = 1$)

value at the end of the investment horizon T. Finally, Section 5 concludes and summarises.

2 Background

2.1 Lévy Processes

Suppose $\phi(u)$ is the characteristic function related to some distribution function. If for every positive integer n, $\phi(u)$ is also the nth power of a characteristic function, we say that the distribution is infinitely divisible.

One can define for every such infinitely divisible distribution a stochastic process $\{X_t, t \geq 0\}$, called a Lévy process, which starts at zero, has independent and stationary increments and such that the distribution of an increment over $[s, s+t]$, $s, t \geq 0$, i.e. $X_{t+s} - X_s$, has $(\phi(u))^t$ as its characteristic function.

The cumulant chacteristic function function $\psi(u) = \ln \phi(u)$ is often called the *characteristic exponent* and it satisfies the following *Lévy-Khintchine formula* :

$$\psi(u) = \mathrm{i}\gamma u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{+\infty}(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux1_{\{|x|<1\}})\nu(\mathrm{d}x), \tag{3}$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with

$$\int_{-\infty}^{+\infty} \inf\{1, x^2\}\nu(\mathrm{d}x) = \int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty.$$

We then say that our infinitely divisible distribution has a triplet of Lévy characteristics (or Lévy triplet for short) $[\gamma, \sigma^2, \nu(\mathrm{d}x)]$. The measure ν is called the *Lévy measure* of X.

From the Lévy-Khintchine formula, one can easily derive that, in general, a Lévy process consists of three independent parts: a linear deterministic part, a Brownian part, and a pure jump part. The Lévy measure ν dictates how the jumps occur. In Table 1 we summarise the Lévy measure for some popular Lévy processes.

Further, for $\mathbf{t} = (t_1, t_2, ..., t_n)$ and $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ let $F_{\mathbf{t}}(\mathbf{x})$ denote the multivariate distribution function of the random vector $(X_{t_1}, X_{t_2}, ..., X_{t_n})$:

$$F_{\mathbf{t}}(\mathbf{x}) = \Pr(X_{t_1} \leq x_1, X_{t_2} \leq x_2, ..., X_{t_n} \leq x_n). \tag{4}$$

When $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is a continuous random vector we denote its density by $f_{\mathbf{t}}(\mathbf{x})$

$$f_{\mathbf{t}}(\mathbf{x}) = \frac{\partial^n}{\partial x_1 \partial x_2 ... \partial x_n} F_{\mathbf{t}}(\mathbf{x}). \tag{5}$$

On the other hand, when $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ is discrete we define $f_{\mathbf{t}}(\mathbf{x})$ as

$$f_{\mathbf{t}}(\mathbf{x}) = \Pr(X_{t_1} = x_1, X_{t_2} = x_2, ..., X_{t_n} = x_n). \tag{6}$$

Finally let $m_t(u)$ denote the moment generating function (mgf) of X_t. We have that $m_t(u) = (m_1(u))^t$ and we will use the short-hand notation $m(u)$ instead of $m_1(u)$.

2.2 Ordering of Risks

Definition 1 ***(Convex Ordering of pay-offs)*** *The pay-off P_g is said to precede the pay-off P_h in the convex order sense, written as $P_g \leq_{cx} P_h$, if the following conditions hold:*

$$\begin{aligned} E_{\mathbb{P}}\left[P_g\right] &= E_{\mathbb{P}}\left[P_h\right], \\ E_{\mathbb{P}}\left[(P_g - d)_+\right] &\leq E_{\mathbb{P}}\left[(P_h - d)_+\right], \quad \textit{for } d \in \mathbb{R}, \end{aligned} \tag{7}$$

In general, the pay-off P_g will depend also on intermediate prices $S(t_i)$ with $0 < t_i < T$. Consider now the conditional expectation $E_{\mathbb{P}}\left[P_g \mid S_T = s\right]$ which can be interpreted as the $\mathbb{P}$-weighted average of P_g, given that the final price S_T of the underlying stock equals s. Then, we let s vary and we obtain the random variable $E_{\mathbb{P}}\left[P_g \mid S_T\right]$. By construction we have that $E_{\mathbb{P}}\left[P_g \mid S_T\right]$ is a function of the final stock price S_T only, and hence is path-independent. Note that $E_{\mathbb{P}}\left[P_g \mid S_T\right]$ meets the requirements of definition (2) and has the same expectation as P_g.

The following result was proven in Kaas *et al.* (2000) and is essentially an application of Jensen's inequality.

Theorem 2 ***(Convex Ordering for Conditional Expectations)*** *Using the notation above we have that*

$$E_{\mathbb{P}}\left[P_g \mid S_T\right] \leq_{cx} P_g. \tag{8}$$

It is well-known that in both von Neumann & Morgenstern's 'Expected Utility Theory' as well as in Yaari's 'Dual Theory of Choice under Risk', convex ordering represents the common preferences of risk averse decision makers with regards to risks with equal expectations; see for example Wang & Young (1998) for more information. From the theorem above we conclude that the pay-off

$E_{\mathbb{P}}\left[P_g \mid S_T\right]$ will dominate the pay-off P_g from the point of view of all risk averse decision makers.

Hence, a risk averse decision maker who has to choose between the path-independent $E_{\mathbb{P}}\left[P_g \mid S_T\right]$ and the path-dependent P_g will always take $E_{\mathbb{P}}\left[P_g \mid S_T\right]$ when the prices $C(E_{\mathbb{P}}\left[P_g \mid S_T\right])$ and $C(P_g)$ are equal.

2.3 Financial Pricing using the Esscher Transform

The pay-offs P_g as defined in (2) depend on the dynamics of the stochastic process $\{S_t,\ t \geq 0\}$. It is well-known that the absence of arbitrage opportunities essentially amounts to determining the price $C(P_g)$ for P_g by taking the discounted expectation of P_g, not with respect to the (initial) physical probability measure $\mathbb{P}$, but with respect to another probability measure $\mathbb{Q}$. We will use the notation $E_{\mathbb{Q}}$ when expectations are taken with respect to this new probability measure $\mathbb{Q}$. Furthermore, $\mathbb{Q}$ has to be determined such that the discounted process $\{e^{-rt}S_t,\ t \geq 0\}$ becomes a martingale, which means that for all $t \geq 0$:

$$E_{\mathbb{Q}}\left[e^{-rt}S_t\right] = S_0, \qquad t \geq 0. \tag{9}$$

We refer to e.g. Harrison & Kreps (1979) or Harrison & Pliska (1981) for extensive theory on arbitrage-free pricing. We conclude that the price $C(P_g)$ of the financial pay-off P_g is given by

$$C(P_g) = e^{-rt}E_{\mathbb{Q}}\left[P_g\right], \tag{10}$$

for some martingale measure $\mathbb{Q}$, and we will now show how the so-called Esscher transform can be used in deriving $\mathbb{Q}$.

The Esscher transform with parameter h of a continuous stochastic process $\{X_t, t \geq 0\}$ is the process where for $t > 0$ the modified probability density function $f_t^{(h)}(x)$ of X_t is defined as:

$$f_t^{(h)}(x) = \frac{e^{hx}f_t(x)}{m_t(h)}, \tag{11}$$

where $h \in \mathbb{R}$. We will denote the modified mgf of X_t by $m_t^{(h)}$. The Esscher transform effectively modifies the initial probability measure $\mathbb{P}$ of the process. Note that since the exponential function is positive, the modified probability measure is equivalent to the physical probability measure and and also that when $h = 0$ we obtain the original probability measure.

The Esscher transform as such has a long history in the actuarial science literature and in mathematical finance. It was first introduced in Esscher (1932) and it was used in actuarial science by Gerber & Shiu (1994a) for the pricing of financial pay-offs. They have shown that when the price follows an exponential Lévy process it can always be used, in the absence of arbitrage opportunities, to construct a (not necessarily unique) equivalent martingale measure.

More precisely, following Gerber & Shiu (1994a) we seek $h = h^*$ such that the discounted stock price process $\{e^{-rt}S_t, t \geq 0\}$ is a martingale with respect to the probability measure $\mathbb{Q}$ that is induced by the parameter h^*.

Distribution	Esscher transformed distribution
Poisson(λ)	Poisson($\exp(h)\lambda$)
Normal(μ, σ^2)	Normal($\mu + \sigma^2 h, \sigma^2$)
Gamma(a, b)	Gamma($a, b - h$)
IG(c, λ)	IG($c, \lambda - h$)
VG(C, G, M)	VG($C, G + h, M - h$)
NIG(α, β, δ)	NIG($\alpha, \beta + h, \delta$)
CGMY(C, G, M, Y)	CGMY($C, G + h, M - h, Y$)
Meixner(α, β, δ)	Meixner($\alpha, \alpha h + \beta, \delta$)

Table 2: The Esscher transform with parameter h of some popular distributions

From (9) and (11) it follows that this corresponds to solving the equation $e^{rt} = m_t^{(h^*)}(1)$. Since it holds that $m_t^{(h)}(x) = \left(m^{(h)}(x)\right)^t$ we can write the equation for h^* as follows:

$$r = \ln(m^{(h^*)}(1)). \tag{12}$$

From Gerber and Shiu (1994b) we observe that h^* is always unique, and also that in the case of a Brownian motion there is only one equivalent martingale measure possible in which case we call the market complete. For general Lévy processes the market will not be complete and the Esscher transform is only one of the methods that can be used to perform arbitrage-free pricing. Note however that the use of the Esscher transform to perform arbitrage-free pricing is also supported using arguments that stem from maximising utility or minimising entropy; see Gerber & Shiu (1994a), Chan (1999) and Raible(2000).

We finally note that if ϕ is the characteristic function and $[\gamma, \sigma^2, \nu(\mathrm{d}x)]$ is the Lévy triplet of X_1, then the characteristic function of X_1 under the Esscher transformed measure will be denoted by $\phi^{(h)}$ and is is given as

$$\ln \phi^{(h)}(u) = \ln \phi(u - \mathrm{i}h) - \ln \phi(-\mathrm{i}h). \tag{13}$$

Moreover this law remains infinitely divisible and its Lévy triplet $[\gamma^{(h)}, (\sigma^{(h)})^2, \nu^{(h)}(\mathrm{d}x)]$ is given by

$$\begin{aligned} \gamma^{(h)} &= \gamma + \sigma^2 h + \int_{-1}^{1} (\exp(hx) - 1)\nu(\mathrm{d}x) \\ \sigma^{(h)} &= \sigma \\ \nu^{(h)}(\mathrm{d}x) &= \exp(hx)\nu(\mathrm{d}x). \end{aligned} \tag{14}$$

From (13) it becomes straightforward to derive the effect of applying the Esscher transform on the distributions that we mentioned previously in Table 1, and we show the results in Table 2. Notice from Table 2 that not all parameters of the distributions will necessarily change when applying the Esscher transform.

3 Inefficiency of Path-dependent Pay-offs

3.1 Main Result

In Section 2 it was shown that $E_{\mathbb{P}}\left[P_g \mid S_T\right] \leq_{cx} P_g$. So, if we can show that the costs (or prices) of $E_{\mathbb{P}}\left[P_g \mid S_T\right]$ and P_g are equal then any risk averse investor will always opt for the path-independent pay-off $E_{\mathbb{P}}\left[P_g \mid S_T\right]$.

We note that in order to show that the financial prices $C(E_{\mathbb{P}}\left[P_g \mid S_T\right])$ and $C(P_g)$ are equal it is sufficient that

$$E_{\mathbb{P}}\left[P_g \mid S_T\right] \equiv E_{\mathbb{Q}}\left[P_g \mid S_T\right], \tag{15}$$

because in this case

$$\begin{aligned} E_{\mathbb{Q}}\left[P_g\right] &= E_{\mathbb{Q}}[E_{\mathbb{Q}}\left[P_g \mid S_T\right]] \\ &= E_{\mathbb{Q}}[E_{\mathbb{P}}\left[P_g \mid S_T\right]]. \end{aligned} \tag{16}$$

Furthermore, (15) will hold if for all $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{t} = (t_1, t_2, ..., t_n) \in \mathbb{R}^n$, and $y \in \mathbb{R}$ we have that:

$$f_{\mathbf{t}}(\mathbf{x} \mid X_T = y) = f_{\mathbf{t}}^{(h^*)}(\mathbf{x} \mid X_T = y). \tag{17}$$

Now, since $\{X_t, t \geq 0\}$ is a Lévy process it follows that:

$$\begin{aligned} f_{\mathbf{t}}(\mathbf{x} \quad &| \quad X_T = y) \\ &= f_{t_1,t_2,...,t_n}(x_1, x_2, ..., x_n \mid X_T = y) \\ &= f_{t_1,t_2-t_1,...,t_n-t_{n-1}}(x_1, x_2 - x_1, ..., x_n - x_{n-1} \mid X_T = y) \\ &= f_{t_1}(x_1 \mid X_T = y) \times f_{t_2-t_1}(x_2 - x_1 \mid X_T = y) \times ... \\ &\quad \times f_{t_n-t_{n-1}}(x_n - x_{n-1} \mid X_T = y). \end{aligned} \tag{18}$$

Hence, it will follow eventually that the financial prices $C(E_{\mathbb{P}}\left[P_g \mid S_T\right])$ and $C(P_g)$ are equal if for all real x, y and $t > 0$ the following is true:

$$f_t(x \mid X_T = y) = f_t^{(h^*)}(x \mid X_T = y), \tag{19}$$

and this will be proven in the next theorem.

Theorem 3 ***(Esscher transform does not change the conditional density)*** *We have that* $f_t(x \mid X(T) = y) = f_t^{(h^*)}(x \mid X(T) = y)$.

Proof.

$$\begin{aligned}
f_t^{(h^*)}(x \quad | \quad & X_T = y) \\
&= \frac{f_{t,T-t}^{(h^*)}(x, y-x)}{f_T^{h^*}(y)} \\
&= \frac{f_t^{(h^*)}(x) \cdot f_{T-t}^{(h^*)}(y-x)}{f_T^{(h^*)}(y)} \\
&= \frac{f_t(x) \cdot \mathrm{e}^{h^* x} \cdot f_{T-t}(y-x) \cdot \mathrm{e}^{h^* y - h^* x}}{f_T(y) \cdot \mathrm{e}^{h^* y}} \\
&\quad \cdot \frac{m_T(h^*)}{m_t(h^*) \cdot M_{T-t}(h^*)} \\
&= \frac{f_t(x) \cdot f_{T-t}(y-x)}{f_T(y)} \\
&= f_t(x \mid X_T = y) \qquad (20)
\end{aligned}$$

■

The above reasoning shows that in an arbitrage-free Lévy market setting, path-dependent financial structures can be outperformed by path-independent structures, at least from the point of view of risk averse decision makers with a fixed horizon and when using the Esscher transform as the pricing rule.

3.2 Example: Inefficiency of Geometric Averaging

The following example explicitly verifies that path-dependent strategies can be dominated by path-independent strategies without increasing the cost and as such confirms the theoretical results.

We will assume that the stochastic process $\{X_t, t \geq 0\}$ is a Brownian motion and we consider $S_t = S_0 e^{X_t}$ with $0 < t \leq 2$ and $S_0 = 1$. Let us define the path-dependent pay-off P_g given by

$$\begin{aligned}
P_g &= (S_1 S_2)^{\frac{1}{2}} \\
&= e^{Z_1 + \frac{Z_2}{2}}, \qquad (21)
\end{aligned}$$

where the random variables $Z_1 = X_1$ and $Z_2 = X_2 - X_1$ are independent and normally $N(\mu, \sigma^2)$ distributed log-returns over the periods $[0, 1[$ and $[1, 2[$, respectively. Note that the random variable P_g represents a geometric average and is lognormally distributed with parameters $\frac{3}{2}\mu$ and $\frac{5}{4}\sigma^2$. Then, we consider the path-independent conditional expectation $E_{\mathbb{P}}[P_g \mid S_2]$. We find that

$$\begin{aligned}
E_{\mathbb{P}}[P_g \mid S_2] &= E_{\mathbb{P}}[P_g \mid X_2] \\
&= E_{\mathbb{P}}[P_g \mid Z_1 + Z_2] \\
&= E_{\mathbb{P}}\left[e^{Z_1 + \frac{Z_2}{2}} \mid Z_1 + Z_2\right].
\end{aligned}$$

From the properties of the bivariate normal random vector $(Z_1 + \frac{Z_2}{2}, Z_1 + Z_2)$ it follows that

$$\begin{aligned} E_{\mathbb{P}}\left[P_g \mid S_2\right] &= e^{\frac{3}{4}(Z_1+Z_2)+\frac{5}{80}\sigma^2} \\ &= (S_2)^{\frac{3}{4}} \mathrm{e}^{\frac{5}{80}\sigma^2}, \end{aligned}$$

Hence $E_{\mathbb{P}}\left[P_g \mid S_2\right]$ is also lognormally distributed but now with parameters $\frac{3}{2}\mu + \frac{5}{80}\sigma^2$ and $\frac{9}{8}\sigma^2$.

Comparing the set of parameters $(\frac{3}{2}\mu + \frac{5}{80}\sigma^2, \frac{9}{8}\sigma^2)$ with $(\frac{3}{2}\mu, \frac{5}{4}\sigma^2)$ it is intuitively clear that, as long as the costs are equal, the pay-off $E_{\mathbb{P}}\left[P_g \mid S(2)\right]$ will be preferred over the pay-off P_g by all risk averse decision makers. This intuition can be explicitly verified by comparing the respective stop-loss premiums, which is left as an easy exercise, or by relying on Theorem 2 directly.

We will now demonstrate that the cost $C(E_{\mathbb{P}}\left[P_g \mid S_2\right])$ of the dominating pay-off $E_{\mathbb{P}}\left[P_g \mid S_2\right]$ is indeed equal to the cost $C(P_g)$ of the pay-off P_g. First, note that under the Esscher equivalent measure $\mathbb{Q}$ we have that P_g is lognormally distributed but now with parameters $\frac{3}{2}(r - \frac{1}{2}\sigma^2)$ and $\frac{5}{4}\sigma^2$, and also

$$\begin{aligned} C(P_g) &= \mathrm{e}^{-2r}(\mathrm{e}^{\frac{3}{2}(r-\frac{1}{2}\sigma^2)+\frac{5}{8}\sigma^2}) \\ &= \mathrm{e}^{-\frac{1}{2}r-\frac{1}{8}\sigma^2}. \end{aligned} \tag{22}$$

$E_{\mathbb{P}}\left[P_g \mid S_2\right]$ is lognormally distributed with parameters $\frac{3}{2}\mu + \frac{5}{80}\sigma^2$ and $\frac{9}{8}\sigma^2$. Consequently we have that

$$\begin{aligned} C(E_{\mathbb{P}}\left[P_g \mid S_2\right]) &= \mathrm{e}^{-2r}(\mathrm{e}^{\frac{3}{2}(r-\frac{1}{2}\sigma^2)+\frac{5}{80}\sigma^2+\frac{9}{16}\sigma^2}) \\ &= \mathrm{e}^{-\frac{1}{2}r-\frac{1}{8}\sigma^2}. \end{aligned} \tag{23}$$

and this shows that $E_{\mathbb{P}}\left[P_g \mid S_2\right]$ does indeed have the same cost as P_g

4 Optimal Path-Independent Pay-offs

4.1 Main Result

In the previous section it was shown that any path-dependent pay-off can be dominated by a path-independent pay-off, and this raises the question whether the broad class of all path-independent pay-offs can be narrowed further. In this section we prove that the 'best' path-independent pay-off $P_g = g(S_T)$ is when g is non-decreasing. The main idea here is that we can keep the same physical distribution for P_g by re-assigning pay-off values to certain realised prices of the underlying stock. This will keep the distribution function of P_g under the initial measure $\mathbb{P}$ unchanged but it will decrease the cost. In order to have a stock market that makes sense economically we need to assume that $E_{\mathbb{P}}[S_t] > \mathrm{e}^{rt}$ and from (11) and (12) it follows that this will imply that $h^* < 0$.

Theorem 4 ***(Optimal path-independent payoffs)*** *For a path-independent investment pay-off P_g to be optimal it must be increasing in the underlying asset value S_T.*

Proof. We will assume that S_T is a discrete and finite random variable that takes values s_i, $(i = 1, 2, ..., n)$ and we denote $\Pr(S_T = s_i) = p_i$. The general case will then follow by taking the appropriate limits. Each realisation s_i for the stock price corresponds to a realisation v_i for the pay-off P_g. Let us now assume that there exist realisations $s_i < s_j$ such that $v_i > v_j$. We will prove that we can find another pay-off with the same physical distribution as P_g but at a lower cost. Without loss of generality we can assume that $i = 1, j = 2$ and also that $n = 2$.

Hence we will assume that $v_1 > v_2$. Consider the pair of Esscher transformed probabilities given by

$$\left(\frac{p_1 \mathrm{e}^{h^* s_1}}{p_1 \mathrm{e}^{h^* s_1} + p_2 \mathrm{e}^{h^* s_2}}, \frac{p_2 \mathrm{e}^{h^* s_2}}{p_1 \mathrm{e}^{h^* s_1} + p_2 \mathrm{e}^{h^* s_2}}\right). \tag{24}$$

We will first consider the case where $p_1 = p_2$. Then the price $C(P_g)$ at time 0 is given by

$$C(P_g) = \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} \left[v_1 \mathrm{e}^{h^* s_1} + v_2 \mathrm{e}^{h^* s_2}\right]. \tag{25}$$

If we switch the two outcomes then the pay-off P_g will change, but not its distribution function, and we will denote the new pay-off by P_h. Whereas the physical probability distribution functions for P_g and P_h coincide, the price will change to $C(P_h)$ given by

$$C(P_h) = \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} \left[v_2 \mathrm{e}^{h^* s_1} + v_1 \mathrm{e}^{h^* s_2}\right]. \tag{26}$$

Comparing $C(P_g)$ and $C(P_h)$ we see that:

$$\begin{aligned} C(P_g) - C(P_h) &= \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} \left[v_1 \mathrm{e}^{h^* s_1} + v_2 \mathrm{e}^{h^* s_2} - v_2 \mathrm{e}^{h^* s_1} - v_1 \mathrm{e}^{h^* s_2}\right] \\ &= \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} (v_1 - v_2) \left(\mathrm{e}^{h^* s_1} - \mathrm{e}^{h^* s_2}\right). \end{aligned} \tag{27}$$

Since $v_1 > v_2$, $s_1 < s_2$ and $h^* < 0$ this is clearly positive and hence we can dominate the original pay-off with one that is increasing with the underlying asset price.

Next we consider the case that $p_1 < p_2$. The proof for $p_1 > p_2$ is similar. The original price is now given by

$$C(P_g) = \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} \left[p_1 v_1 \mathrm{e}^{h^* s_1} + p_2 v_2 \mathrm{e}^{h^* s_2}\right]. \tag{28}$$

We then change the pay-offs so that at the terminal value s_1 there is a pay-off of v_2 with probability p_1. At s_2 there is a p_1 probability of a pay-off of v_1 and a $p_2 - p_1$ probability of a pay-off of v_2. This leaves the physical distribution unchanged. In contrast, the price will now change to:

$$C(P_g) = \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} \left[p_1 v_2 \mathrm{e}^{h^* s_1} + (p_2 - p_1)\, v_2 \mathrm{e}^{h^* s_2} + p_1 v_1 \mathrm{e}^{h^* s_2}\right]. \tag{29}$$

Comparing $C(P_g)$ and $C(P_h)$ we see that:

$$\begin{aligned} C(P_g) - C(P_h) &= \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} \left[p_1 v_1 \mathrm{e}^{h^* s_1} - p_1 v_2 \mathrm{e}^{h^* s_1} - p_1 v_1 \mathrm{e}^{h^* s_2} + p_1 v_2 \mathrm{e}^{h^* s_2} \right] \\ &= \frac{\mathrm{e}^{-rT}}{\mathrm{e}^{h^* s_1} + \mathrm{e}^{h^* s_2}} p_1 \left(v_1 - v_2 \right) \left(\mathrm{e}^{h^* s_1} - \mathrm{e}^{h^* s_2} \right). \end{aligned} \tag{30}$$

Since $v_1 > v_2$, $s_1 < s_2$ and $h^* < 0$ this is positive, and hence we have found another pay-off with the same distribution function under $\mathbb{P}$ but at a lower price. This new pay-off takes values that are increasing with the underlying asset. ■

We note that results in the same vein can already be found in Dybvig (1988b). This author investigated the optimality of investment strategies in any complete market with the objective of determining the strategy with minimal cost whilst preserving a given (physical) probability distribution. In contrast, we examine Lévy markets which are not necessarily complete using the Esscher transform to derive an arbitrage-free price.

4.2 Example: Click fund

In this example we assume a Brownian motion for the stochastic process $\{X_t, t \geq 0\}$, and we consider $S_t = S_0 \mathrm{e}^{X_t}$ with $0 < t \leq 8$ and $S_0 = 1$. Under the physical probability measure $\mathbb{P}$ we have that S_t is lognormally distributed with parameters $(\mu t, \sigma^2 t)$. We also define the indicator random variable I_i as $I_i = 1$ if $S(i) > S(i-1)$ and $I_i = 0$ otherwise. Let us consider the path-dependent pay-off P_g given by

$$P_g = 100(1 + 0.1 \sum_{l=1}^{8} I_i). \tag{31}$$

P_g can be interpreted as follows. There is a guaranteed amount of 100, and for every year that the stock market increases we "click" a bonus of 10. There is no bonus when the stock market declines. Next, after 8 years we take the sum of the bonuses and this will be added to the guaranteed amount. It is easy to see that P_g has the following physical probability function

$$\Pr(P_g = 100 + 10i) = \binom{8}{i} p^i \left(1 - p\right)^{8-i}, \qquad i = 0, 1, ..., 7, 8, \tag{32}$$

where $p = \Pr(N(\mu, \sigma^2) > 0)$. We will denote $\Pr(P_g \leq 100 + 10i)$ by k_i. Furthermore, for its price we find from (10) that

$$C(P_g) = \mathrm{e}^{-8r}(100 + 80q), \tag{33}$$

where $q = \Pr(N(r - \frac{1}{2}\sigma^2, \sigma^2) > 0)$ and r is the yearly risk free rate. In the remainder of the example we will take $\mu = 0.08, \sigma = 0.20$ and r is set at 0.045. Using our parameter values we find that $p \approx 0.6554$, $q \approx 0.5497$. and also that $\mathrm{C}(P_g) \approx 100.45$. Then, we will consider another pay-off P_h given by

$$P_h = 100(1 + 0.1 \sum_{l=1}^{8} J_i). \tag{34}$$

Here J_i is an indicator random variable that takes the value 1 if $S_8 > \alpha_i$, $(i = 1, 2, ..., 8)$ with $\alpha_i = \mathrm{e}^{8\mu+\sqrt{8\sigma^2}\phi^{-1}(k_{i-1})}$, where ϕ^{-1} denotes the quantile function of the standard normal random variable. It is easy to verify that under the physical measure $\mathbb{P}$, the pay-off P_h has the same distribution function as P_g. On the other hand for the price $C(P_h)$ we find that

$$C(P_h) = 100\mathrm{e}^{-8r}(1 + 0.1 \sum_{l=1}^{8} \Pr(J_i = 1)), \tag{35}$$

where $\Pr(J_i = 1)$ now denotes the probability under $\mathbb{Q}$ that $S_8 > \alpha_i$. It is easily verified that $C(P_h) \approx 99.05$. Hence we have constructed a pay-off P_h that under the probability measure $\mathbb{P}$ has the same distribution function as the pay-off P_g of the click-fund, but at the lowest possible cost.

5 Conclusion

In this paper we have examined the optimality of investment pay-offs in Lévy markets under the risk-neutral Esscher martingale measure. We provide a simple proof for the Cox & Leland result that in a Black & Scholes market risk averse decision makers prefer path-independent strategies over path-dependent strategies and we extend their results to general Lévy markets. Furthermore, optimal path-independent pay-offs are those which are increasing with the underlying asset value - a result that is closely related to the results of Dybvig (1988a,b). These results imply that path-dependent investment pay-offs, the use of which is widespread in financial markets, do not appear to offer good value for risk averse decision makers with a fixed investment horizon.

This observation holds in a Black & Scholes market, the use of which may be justified when the investment horizon is longer than one year and is also true for general Lévy markets when arbitrage-free pricing is performed using Esscher transforms. We remark that the use of the Esscher transform as the pricing rule is also supported using arguments that stem from maximising utility or minimising entropy; see Gerber & Shiu (1994a), Chan (1999) and Raible(2000).

When deriving our results we assumed perfect markets in particular excluding the impact of transaction costs, liquidity aspects and presence of asymmetric information. It is easily seen that besides their intrinsic superiority for utility maximisers path-independent structures are also preferable for liquidity reasons. Indeed, a path-independent pay-off with a given maturity T may be approximated by a combination of a zero coupon bonds and a series of single call options. Such a portfolio does not require intermediate trading and is immune to liquidity risk during the horizon T of the product. In future research we will also focus on the impact of transaction costs.

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